THE SUBGROUP DETERMINED BY A CERTAIN IDEAL IN A FREE GROUP RING

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Abstract. For normal subgroups R and S of a free group F, an identification of the subgroup $F\cap(1+\mathfrak{rfs})$ is derived, and it is shown that the the quotient $\frac{F\cap(1+\mathfrak{rfs})}{[R'\cap S',\, R\cap S][R'\cap S,\, R'\cap S][R\cap S',\, R\cap S']}$ is, in general, non-trivial.

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1. Introduction

Every two-sided ideal $\mathfrak{a}$ in the integral group ring $\mathbb{Z}[F]$ of a free group F determines a normal subgroup $F\cap(1+\mathfrak{a})$ of F. Identification of such subgroups is a fundamental problem in the theory of group rings ([5], [9]). Let R and S be normal subgroups of F. In this paper we examine the subgroup $F\cap(1+\mathfrak{rfs})$, where, for a normal subgroup G of F, $\mathfrak{g}$ denotes the two-sided ideal of $\mathbb{Z}[F]$ generated by $G-1$. This subgroup has been studied by C. K. Gupta [4] (see also [7]). It is easy to check that

$$[R'\cap S',\, R\cap S][R'\cap S,\, R'\cap S][R\cap S',\, R\cap S']\subseteq F\cap(1+\mathfrak{rfs}),$$

where R' (resp. S') is the derived subgroup of R (resp. S). Whereas the identification given in [4], namely that the preceding inclusion is an equality, holds up to torsion, our investigation shows that, $\frac{F\cap(1+\mathfrak{rfs})}{[R'\cap S',\, R\cap S][R'\cap S,\, R'\cap S][R\cap S',\, R\cap S']}\cong L_1\mathsf{SP}^2\left(\frac{R\cap S}{(R'\cap S)(R\cap S')}\right)$, and is, in general, non-identity; here $L_1\mathsf{SP}^2$ is the first derived functor of the second symmetric power functor.

2. The subgroup $F\cap(1+\mathfrak{rfs})$

Let F be a free group and R, S its normal subgroups with bases, as free groups, $\{r_i\}_{i\in I}$ and $\{s_j\}_{j\in J}$ respectively. Then the ideal $\mathfrak{r}$ is a free right $\mathbb{Z}[F]$-module with basis $\{r_i-1\,|\,i\in I\}$ and the ideal $\mathfrak{s}$ is a free left $\mathbb{Z}[F]$-module with basis $\{s_j-1\,|\,j\in J\}$ ([3], Theorem 1, p. 32). Further, recall that

$$R/R'\cong\frac{\mathfrak{r}}{\mathfrak{rf}}\quad(\cong\frac{\mathfrak{r}}{\mathfrak{fr}})$$

this isomorphism being given by

$$rR'\mapsto(r-1)+\mathfrak{rf},\ r\in R\quad(\text{resp. } rR'\mapsto r-1+\mathfrak{fr}).$$

From these observations it immediately follows that we can make the following identification

$$R/R' \otimes S/S' = \frac{\mathfrak{r}}{\mathfrak{r}\mathfrak{f}} \otimes \frac{\mathfrak{s}}{\mathfrak{f}\mathfrak{s}} = \frac{\mathfrak{r}}{\mathfrak{r}\mathfrak{f}} \otimes_{\mathbb{Z}[F]} \frac{\mathfrak{s}}{\mathfrak{f}\mathfrak{s}} = \frac{\mathfrak{r}\mathfrak{s}}{\mathfrak{r}\mathfrak{f}\mathfrak{s}}. \tag{2.1}$$

Here $\otimes$ is tensor product over $\mathbb{Z}$ which we can replace by $\otimes_{\mathbb{Z}[F]}$ since the action of $\mathbb{Z}[F]$ on components of the tensor product is trivial.

Theorem 2.1. *If R and S are normal subgroups of a free group F, then there is a natural isomorphism*

$$\frac{F \cap (1 + \mathfrak{r}\mathfrak{f}\mathfrak{s})}{[R' \cap S', R \cap S][R' \cap S, R' \cap S][R \cap S', R \cap S']} \cong L_1\mathsf{SP}^2\left(\frac{R \cap S}{(R' \cap S)(R \cap S')}\right).$$

Proof. Let us set

$$Q := \frac{R \cap S}{R' \cap S'}, \quad U := \frac{R' \cap S}{R' \cap S'}, \quad V := \frac{R \cap S'}{R' \cap S'}. \tag{2.2}$$

The group Q is free abelian because it injects into $R/R' \oplus S/S'$, and so are U, V both being subgroups of Q. Observe that Q/U is also free abelian, since it is isomorphic to the subgroup $\frac{R \cap S}{R' \cap S}$ of R/R'.

For an abelian group A, we denote by $\mathsf{SP}^2(A)$ its symmetric square, defined as the quotient $\mathsf{SP}^2(A) := A \otimes A / \langle a \otimes b - b \otimes a, \,|\, a, b \in A\rangle$ and by $\Lambda^2(A)$ its exterior square $\Lambda^2(A) := A \otimes A / \langle a \otimes a \,|\, a \in A\rangle$. Recall (see [8]) that, for any free resolution

$$0 \to C \to B \to A \to 0$$

of A, the so-called *Koszul complex*

$$0 \to \Lambda^2(C) \to C \otimes B \to \mathsf{SP}^2(B)$$

represents the object $L\mathsf{SP}^2(A)$ of the derived category of abelian groups; in particular, its zeroth (resp. first) homology is equal to the zeroth (resp. first) derived functor of SP^2 applied to A.

Consider the natural commutative diagram with exact rows and columns which contains maps between quadratic Koszul complexes:

$$\begin{tikzcd}
\Lambda^2(U) \arrow[r, tail] \arrow[d, tail] & U \otimes Q \arrow[r] \arrow[d, tail] & \mathsf{SP}^2(Q) \arrow[d, equal] \\
\Lambda^2(Q) \arrow[r] \arrow[d] & Q \otimes Q \arrow[r] \arrow[d] & \mathsf{SP}^2(Q) \\
\frac{\Lambda^2(Q)}{\Lambda^2(U)} \arrow[r] & Q/U \otimes Q &
\end{tikzcd}$$

Since the middle horizontal complex is acyclic, the homology of the lower complex are the same as of the upper complex shifted by one. That is, there exists a short exact sequence

$$0 \to \frac{\Lambda^2(Q)}{\Lambda^2(U)} \to Q/U \otimes Q \to \mathsf{SP}^2(Q/U) \to 0$$

which can be naturally extended to the following diagram:

$$\begin{tikzcd}
 & & & K \arrow[d, tail] \\
 & & Q/U \otimes V \arrow[r, equal] \arrow[d, tail] & Q/U \otimes V \arrow[d] \\
 & \frac{\Lambda^2(Q)}{\Lambda^2(U)} \arrow[r, tail] \arrow[d, equal] & Q/U \otimes Q \arrow[r, two heads] \arrow[d] & \mathsf{SP}^2(Q/U) \\
K \arrow[r, tail] & \frac{\Lambda^2(Q)}{\Lambda^2(U)} \arrow[r] & Q/U \otimes Q/V &
\end{tikzcd} \tag{2.3}$$

Here K is, by definition, the kernel of the lower horizontal map. By Snake Lemma, K is isomorphic to the kernel of the right hand vertical map $Q/U \otimes V \to \mathsf{SP}^2(Q/U)$ in the diagram. Observe that this map is part of the Koszul complex

$$0 \to \Lambda^2(VU/U) \to Q/U \otimes V \to \mathsf{SP}^2(Q/U)$$

which represents the object $L\mathsf{SP}^2(Q/UV)$ of the derived category of abelian groups. Here we have used the fact that $V = VU/U = V/(V \cap U)$, since $V \cap U$ is the zero subgroup of Q. The homology groups of the above Koszul complex are the derived functor evaluations $L_i\mathsf{SP}^2(Q/UV),\ i = 1,\, 2$ (see [8]). Therefore, we get the following short exact sequence:

$$0 \to \Lambda^2(V) \to K \to L_1\mathsf{SP}^2(Q/UV) \to 0.$$

Consequently the lower sequence of the diagram (2.3), yields the following exact sequence:

$$0 \to L_1\mathsf{SP}^2(Q/UV) \to \frac{\Lambda^2(Q)}{\Lambda^2(U) + \Lambda^2(V)} \to Q/U \otimes Q/V \tag{2.4}$$

We next observe that there are natural isomorphisms

$$\Lambda^2(Q) \cong \frac{\gamma_2(R \cap S)}{[R' \cap S',\, R \cap S]}$$

$$\frac{\Lambda^2(Q)}{\Lambda^2(U) + \Lambda^2(V)} \cong \frac{\gamma_2(R \cap S)}{[R' \cap S',\, R \cap S][R' \cap S,\, R' \cap S][R \cap S',\, R \cap S']},$$

and natural monomorphisms $Q/U \to R/R'$, $Q/V \to S/S'$. The exact sequence (2.4) thus implies that there is an exact sequence

$$0 \to L_1\mathsf{SP}^2(Q/UV) \to \frac{\gamma_2(R \cap S)}{[R' \cap S', R \cap S][R' \cap S, R' \cap S][R \cap S', R \cap S']} \to R/R' \otimes S/S'. \tag{2.5}$$

The statement of the theorem follows from the fact (see [2]) that

$$F \cap (1 + \mathfrak{r}\mathfrak{s}) = \gamma_2(R \cap S)$$

and the identification (2.1). $\square$

For an abelian group A, a description of the group $L_1\mathsf{SP}^2(A)$ is available in many papers on polynomial functors; for example, see [1] or ([6], Theorem 2.2.5). Recall the main properties of $L_1\mathsf{SP}^2(A)$. For any abelian group A, $L_1\mathsf{SP}^2(A)$ is a natural quotient of the group $\mathsf{Tor}(A,\, A)$ by diagonal elements. We have

$$L_1\mathsf{SP}^2(\mathbb{Z}/m\mathbb{Z}) = L_1\mathsf{SP}^2(\mathbb{Z}) = 0,$$

for all natural numbers m, and, for all abelian groups A, B, there is a (bi)natural isomorphism

$$\mathsf{Tor}(A,\, B) = \mathsf{Ker}\{L_1\mathsf{SP}^2(A \oplus B) \twoheadrightarrow L_1\mathsf{SP}^2(A) \oplus L_1\mathsf{SP}^2(B)\}.$$

For a free abelian group A and a natural number $m \geq 1$, there is a natural isomorphism

$$L_1\mathsf{SP}^2(A \otimes \mathbb{Z}/m\mathbb{Z}) \simeq \Lambda^2(A \otimes \mathbb{Z}/m\mathbb{Z}).$$

Observe also that, the functor $L_1\mathsf{SP}^2$ is related to the homology of the Eilenberg-MacLane spaces $K(-,\, 2)$. Namely, for any abelian group A, there is a natural short exact sequence

$$0 \to L_1\mathsf{SP}^2(A) \to H_5K(A,\, 2) \to \mathsf{Tor}(A,\, \mathbb{Z}/2\mathbb{Z}) \to 0.$$

Invoking this description for $L_1\mathsf{SP}^2(Q/UV)$, we have the following identification of the subgroup $F \cap (1 + \mathfrak{r}\mathfrak{f}\mathfrak{s})$:

Theorem 2.2.

$$F \cap (1 + \mathfrak{r}\mathfrak{f}\mathfrak{s}) = [R' \cap S',\, R \cap S][R' \cap S,\, R' \cap S][R \cap S',\, R \cap S']W,$$

where W is the subgroup of F generated by elements[1]

$$[x_1,\, y][x,\, y_2]^{-1},$$

such that

$$\begin{aligned} &x,\, y \in R \cap S,\;\; m \geq 2,\\ &x^m = x_1x_2,\;\; y^m = y_1y_2,\\ &x_1,\, y_1 \in R' \cap S,\\ &x_2,\, y_2 \in R \cap S'. \end{aligned}$$

[1]For an elements g, h of a group, we use the standard commutator notation $[g,\, h] := g^{-1}h^{-1}gh$.

Proof. Consider the generating elements from W, as in the Theorem. Modulo $\mathfrak{rfs}$, we have

$$\begin{aligned}[x_1, y][x, y_2]^{-1} - 1 &\equiv [x^m, y][x_2, y]^{-1}[x, y_2]^{-1} - 1\\ &\equiv (x^m-1)(y-1) - (y^m-1)(x-1)\\ &\quad - (x_2-1)(y-1) + (y-1)(x_2-1)\\ &\quad - (x-1)(y_2-1) + (y_2-1)(x-1)\\ &\equiv (x_1-1)(y-1) - (y_1-1)(x-1) +\\ &\quad (y-1)(x_2-1) - (x-1)(y_2-1).\end{aligned}$$

All four products $(x_1-1)(y-1)$, $(y_1-1)(x-1)$, $(y-1)(x_2-1)$, $(x-1)(y_2-1)$ lie in $\mathfrak{rfs}$. The subgroup W is chosen as a subgroup of representatives of $L_1\mathsf{SP}^2\left(\frac{R\cap S}{(R'\cap S)(R\cap S')}\right)$ in $F\cap(1+\mathfrak{rfs})$.

Consider generators of $L_1\mathsf{SP}^2\left(\frac{R\cap S}{(R'\cap S)(R\cap S')}\right)$ viewed as a natural quotient of the group $\mathsf{Tor}\left(\frac{R\cap S}{(R'\cap S)(R\cap S')}, \frac{R\cap S}{(R'\cap S)(R\cap S')}\right)$. The generators are given as pairs of elements (x, y), $x, y \in R\cap S$, with the property that, there exists $m \geq 2$, such that $x^m, y^m \in (R'\cap S)(R\cap S')$. Consider now the diagram (2.3) and find the image of the pair (x, y) in the quotient $\frac{Q/U\otimes Q}{\Lambda^2(V)}$ (here we use the notation 2.2) and choose its representative in $Q/U\otimes V$. It is given as

$$(x.U)\otimes y_2.(R'\cap S') - (y.U)\otimes x_2.(R'\cap S'), \tag{2.6}$$

where x_2, y_2 are defined in the formulation of the Theorem. Going further in the diagram (2.3), we find a representative of the element (2.6) in $\Lambda^2(Q)/\Lambda^2(V)$, given as

$$(x\wedge y_2) + (x_2\wedge y) - (x^m\wedge y) + \Lambda^2(V).$$

Indeed, the natural map $\Lambda^2(Q)\to Q/U\otimes Q$ sends (we omit the notation $-.(R'\cap S')$ for the elements from Q for the sake of simplification of notations)

$$\begin{aligned}(x\wedge y_2) + (x_2\wedge y) - (x^m\wedge y) \mapsto\ & x.U\otimes y_2 - y_2.U\otimes x + x_2.U\otimes y - y.U\otimes x_2\\ & - x_1x_2.U\otimes y - y.U\otimes x_1x_2 =\\ & x.U\otimes y_2 - y_2.U\otimes x - y.U\otimes x_1 =\\ & x.U\otimes y_2 - y^m.U\otimes x - y.U\otimes x_1 =\\ & x.U\otimes y_2 - y.U\otimes x^m - y.U\otimes x_1 =\\ & x.U\otimes y_2 - y.U\otimes x_2.\end{aligned}$$

In the free group F, this element is represented as a product of commutators

$$[x, y_2][x_2, y][x^m, y]^{-1}.$$

Since modulo $\mathfrak{rfs}$,

$$[x_1, y][x, y_2]^{-1} - 1 \equiv [x^m, y][x_2, y]^{-1}[x, y_2]^{-1} - 1 \equiv ([x, y_2][x_2, y][x^m, y]^{-1})^{-1} - 1$$

we get the asserted description of the set W. □

Remark. Since the groups $F/\gamma_2(R\cap S)$, R/R', S/S' are always torsion-free, the sequence (2.5) implies that there is the following identification

$$L_1\mathsf{SP}^2\left(\frac{R\cap S}{(R'\cap S)(R\cap S')}\right)\cong \textsf{torsion of } \frac{F}{[R'\cap S', R\cap S][R'\cap S, R'\cap S][R\cap S', R\cap S']}.$$

3. Example

Finally, let us give an example of subgroups $R,\ S$ in a free group F, such that

$$L_1\mathsf{SP}^2\left(\frac{R\cap S}{(R'\cap S)(R\cap S')}\right)\neq 0.$$

Let $F=F(a_1,\ \dots\ ,\ a_n,\ b)$, $n\geq 2$,

$$R=\langle a_1,\ \dots\ ,\ a_n,\ [F,\ F]\rangle^F,$$
$$S=\langle a_1^2,\ \dots\ ,\ a_n^2,\ b,\ [F,\ F]\rangle^F.$$

Since $[F,\ F]\subset R$, $[F,\ F]\subset S$,

$$(R'\cap S)(R\cap S')=R'S'.$$

For every $i=1,\ \dots\ ,\ n$, the element $[a_i,\ b]$ lies in $R\cap S$. Observe that,

$$[a_i^2,\ b]=[a_i,\ b][[a_i,\ b],\ a_i][a_i,\ b]$$

Therefore,

$$[a_i,\ b]^2\in R'S'.$$

Since $R'S'=\langle [a_i,\ a_j],\ [a_i,\ b]^2,\gamma_3(F)\rangle$, the elements $[a_i,\ b]$, $i=1,\ \dots\ ,\ n$ form an abelian subgroup of $\frac{R\cap S}{(R'\cap S)(R\cap S')}$ isomorphic to $(\mathbb{Z}/2)^{\oplus n}$. For $n\geq 2$, the first derived functor of SP^2 of such group is non-zero.

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